IN-PLANE PLASTIC LIMIT LOAD OF STEEL CIRCULAR ARCHES
A lower bound limit analysis approach

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INTRODUCTION

This paper describes an analytical method to determine the first-order, in-plane plastic limit load F_{pl} of steel circular arches. In a research project on the out-of-plane stability of steel arches which is being carried out at the Eindhoven University of Technology it is investigated whether this load capacity can be used to develop design rules to predict the strength of free-standing arches with the use of buckling curves. In such design rules the non-dimensional slenderness $\overline{\lambda}$ of the arch plays an important role:

$$\overline{\lambda} = \sqrt{F_{pl} / F_{cr}} \tag{1}$$

where F_{cr} is the out-of-plane buckling load.
Limit analysis of arches is more complex than limit analysis of beams since the influence of normal forces in general cannot be neglected. Chakrabarty [1] describes a method to determine the limit load of arches, based on the work by Onat and Prager [2]. This method requires the interaction curve for combined bending and axial force in the cross-section to be approximated by piecewise linear segments. In this paper a method is described which does not require this approximation. The method will be checked by comparing the limit load determined using this method to results from finite element simulations. The paper is limited to circular arches with either pinned or fixed supports, subjected to a concentrated load F at the top, with a maximum subtended angle 2γ of 180^0 *(Fig. 1)*, failing by an arch mechanism. From the finite element simulations it was found that for small subtended angles the arches fail by a beam mechanism (*Figs. 2* and *3*), as also noted by Stevens [3]. For a beam mechanism in an arch with fixed supports, the outer plastic hinges do not necessarily form at the supports. This is so because the bending moment at the fixed support may be either positive or negative, enabling a gradual transition from arch to beam mechanism.

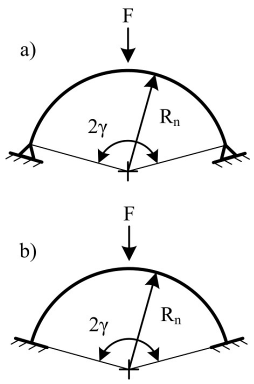


Fig. 1. Considered arches
a) pinned
b) fixed

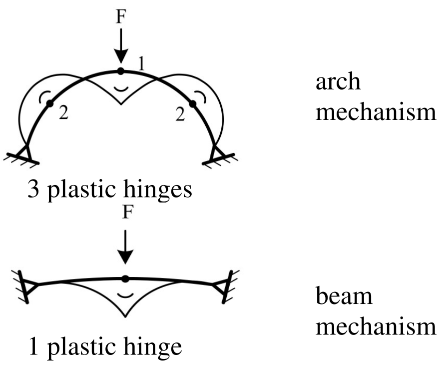


Fig. 2. Bending moment distributions for pinned arch

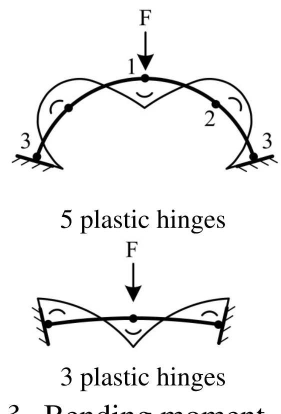


Fig. 3. Bending moment distributions for fixed arch

1 APPROACH

To determine the in-plane limit load of arches an upper-bound and a lower-bound approach can be used. In this study the lower-bound approach has been used, which starts by assuming a moment distribution that is in equilibrium with the external loads, for which the yield condition is nowhere violated. If only bending moments are present, the distribution should satisfy the condition $M \leq M_{pl}$, where M_{pl} is the full plastic moment capacity of the section. If also normal forces are present, the condition $M \leq M_{pl;red}$ should be satisfied, where $M_{pl;red}$ is a reduced plastic moment capacity.

The lower-bound approach generally results in a lower bound of the limit load, unless a moment distribution is assumed which allows the development of a mechanism. It was found that for the check whether a mechanism can develop it is not enough to show that a sufficient number of plastic hinges (defined as points where $M=M_{pl;red}$) is formed; also the kinematic admissibility of the mechanism must be checked.

2 INTERACTION CURVE AND NORMALITY RULE

This paper focuses on arches with rectangular solid cross-sections, for which the influence of the shear force on the plastic moment capacity can be neglected. For such a cross-section the yield condition (*M-N* interaction diagram) is [4]:

$$\psi = M / M_{pl} + \left(N / N_{pl}\right)^2 - 1 = 0 \tag{2}$$

where $M_{pl} = \frac{1}{4} b h^2 f_y$ is the full plastic moment capacity, $N_{pl} = bhf_y$ the full plastic normal force capacity, b the width of the beam, h the height of the beam and f_y the yield stress. From this yield condition the reduced plastic moment capacity $M_{pl;red}$ can be derived as:

$$M_{pl;red} / M_{pl} = 1 - \left(N / N_{pl}\right)^2 \tag{3}$$

In a plastic hinge where not only a bending moment but also a compressive normal force is acting, both a plastic rotation φ and a plastic contraction δ occur. The ratio between φ and δ , which is needed for checking the kinematic admissibility of the mechanism, can be determined from the normality rule:

$$\frac{\delta}{\varphi} = \frac{\partial \psi / \partial N}{\partial \psi / \partial M} = 2 \frac{N}{N_{pl}} \frac{M_{pl}}{N_{pl}} \tag{4}$$

3 EQUILIBRIUM EQUATIONS FOR THE ARCH MECHANISM

Due to symmetry only one half of the arch needs to be considered (*Fig. 4*). It is assumed that plastic hinge no. 1, occurring at $\theta = \theta_1 = 0$, has a reduced plastic moment capacity $M_{pl1;red}$ while plastic hinge no. 2 occurring at $\theta = \theta_2$, has a reduced plastic moment capacity $M_{pl2;red}$. For a fixed arch, plastic hinge no. 3 occurring at $\theta = \gamma$ has a reduced plastic moment capacity $M_{pl3;red}$. The horizontal and vertical reaction forces can be determined from equilibrium conditions, resulting in:

$$R_{BV} = \tfrac{1}{2} F \qquad R_{AH} = R_{BH} = \frac{\tfrac{1}{2} F\, R_n \sin\gamma - M_{pl1;red} + M_{pl3;red}}{R_n \left(1 - \cos\gamma\right)} \tag{5}$$

For a pinned arch $M_{pl3;red}$ should be taken equal to zero in *Eq*. (5) because the moment at the pinned support is zero. The bending moment M and normal force N can then be determined as:

$$M(\theta) = M_{pl1;red} + R_{AH} R_n \left(1 - \cos\theta\right) - \tfrac{1}{2} F\, R_n \sin\theta \qquad N(\theta) = -R_{BH} \cos\theta - R_{BV} \sin\theta \tag{6}$$

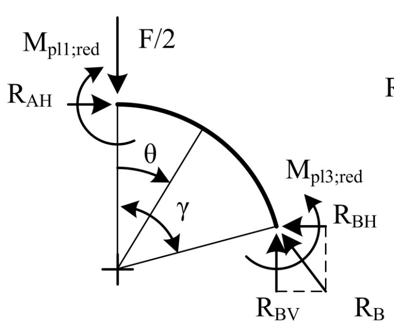


Fig. 4. Geometry and reactions of right half of arch

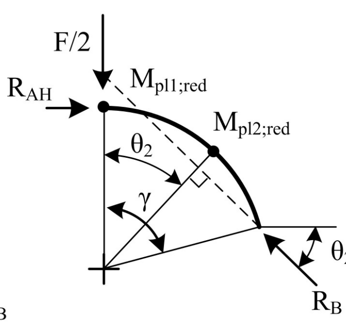


Fig. 5. Funicular line for pinned arch

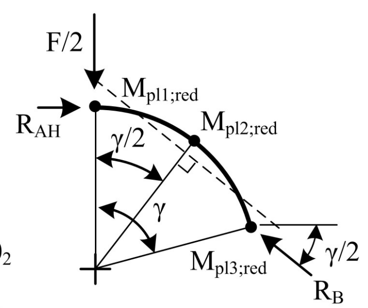


Fig. 6. Funicular line for fixed arch

If $N(\theta)$ is known then $M_{pl;red}(\theta)$ can be determined from *Eq*. (3). In the bending moment distribution described by *Eq*. (6) it is assumed that only the plastic hinges no. 1 and no. 3 have been formed. To determine the limit load F_{pl} for which also plastic hinge no. 2 is formed it is required that:

$$M(\theta=\theta_2) = -M_{pl2;red} \tag{7}$$

Solving F_{pl} from this equation results in:

$$F_{pl} = \frac{2\left(M_{pl2;red} + M_{pl3;red} - \left(M_{pl1;red} + M_{pl2;red}\right)cos\,\gamma + \left(M_{pl1;red} - M_{pl3;red}\right)cos\,\theta_2\right)}{R_n\left(sin(\gamma - \theta_2) - sin\,\gamma + sin\,\theta_2\right)} \tag{8}$$

The limit load F_{pl} given by *Eq*. (8) depends on the angle θ_2 where the plastic hinge no. 2 is formed. For a pinned arch (for which $M_{pl3;red}$ should be taken equal to zero in *Eq*. (8)) this angle can be determined by solving θ_2 from the equation $dF_{pl}/d\theta_2 = 0$ with the help of the program Mathematica [5], resulting in:

$$\theta_2 = \arccos \frac{-M_{pl1;red}M_{pl2;red} + M_{pl1;red}C\cos\gamma + 4\sqrt{M_{pl2;red}}\,C^{3/2}\cos\tfrac{1}{2}\gamma\sin^2\tfrac{1}{2}\gamma}{M_{pl1;red}^2 + 2M_{pl1;red}M_{pl2;red} + 2M_{pl2;red}^2 - 2M_{pl2;red}C\cos\gamma} \tag{9}$$

where:

$$C = M_{pl1;red} + M_{pl2;red} \tag{10}$$

For a fixed arch it can be shown [6] that:

$$M_{pl3;red} = M_{pl1;red} \qquad N_3 = N_1 \qquad \theta_2 = \gamma/2 \tag{11}$$

where N_3 and N_1 are the normal forces in plastic hinges no. 3 respectively no. 1. To determine the reduced plastic moment capacities $M_{pl1;\mathrm{red}}$ and $M_{pl2;\mathrm{red}}$ the normal forces N_1 and N_2 in plastic hinges no. 1 and no. 2 are needed. These forces can be expressed in the support reactions:

$$N_1 = N(\theta = 0) = -R_{AH} = -R_{BH} \tag{12}$$

$$N_2 = N(\theta = \theta_2) = -R_B = -\sqrt{R_{BH}^2 + R_{BV}^2} \tag{13}$$

Eq. (12) can be derived from *Eq*. (6). *Eq*. (13) makes use of the fact that the location of the maximum negative (hogging) moment corresponds to the location of maximum normal force (see *Figs*. *5* and *6*). Because the support reactions depend on the load *F*, the reduced plastic moment capacities depend on the limit load F_{pl} to be determined. Hence an iterative procedure is needed to determine F_{pl}.

4 ITERATIVE PROCEDURE

If we know N_1 we can calculate the reduced plastic moment capacity $M_{pl1;red}$ from the interaction curve (*Eq*. (3)), and the corresponding load *F* from *Eqs*. (12) and (5):

$$F = \frac{2\left(M_{pl1;red} - M_{pl3;red} + N_1R_n\left(cos\,\gamma - 1\right)\right)}{R_n\,sin\,\gamma} \tag{14}$$

For a fixed arch $M_{pl3;red} = M_{pl1;red}$ (see *Eq*. (11)), for a pinned arch $M_{pl3;red}$ should be taken equal to zero in *Eq*. (14), since the moment at the support is zero. Hence we can also calculate the normal force N_2 , using *Eqs*. (13) and (5), and the reduced plastic moment capacity $M_{pl2;\mathrm{red}}$. Then also the limit load F_{pl} can be determined from *Eqs*. (8), (9) and (10).

For the exact value of N_1, $F=F_{pl}$. If N_1 is assumed too small then $F>F_{pl}$ and if N_1 is too large then $F<F_{pl}$. This means that the normal force N_1 can be determined using the bisection method (*Fig*. *7*), where ε is the convergence tolerance. After finishing the iterative procedure it can be checked whether indeed everywhere in the arch: $M(\theta) \le M_{pl;red}(\theta)$.

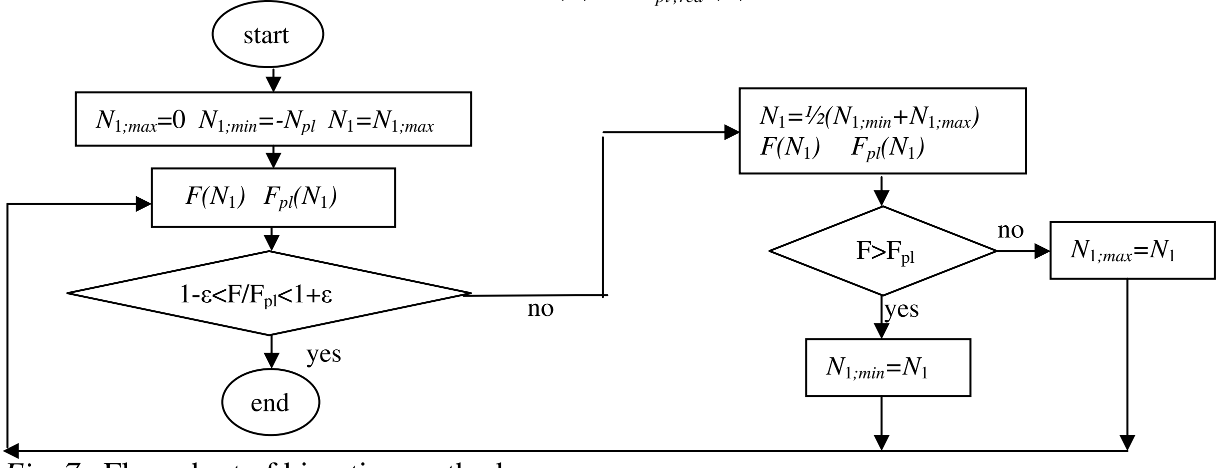


Fig. *7*. Flow chart of bisection method

5 KINEMATIC ADMISSIBILITY OF ARCH MECHANISM

The lower bound approach described in the previous sections ensures that a bending moment distribution can be found with a sufficient number of plastic hinges to enable the development of an arch mechanism. To check whether this arch mechanism can indeed develop, the kinematic admissibility of the mechanism must be checked. *Fig. 8* therefore shows the rotations φ_i and contractions δ_i occurring in the plastic hinges of the arch mechanism, where the subscript *i* denotes the number of the hinge. In a first order limit load calculation the magnitude of the rotations and contractions cannot be determined, only the relation between them. Therefore all rotations and contractions will be expressed in the rotation φ_1.

It was found that for a pinned arch kinematic admissibility requires:

$$\Delta_{ver}/\varphi_1 \geq 0 \tag{15}$$

and:

$$\varphi_2/\varphi_1 \geq 0 \tag{16}$$

where Δ_{ver} is the vertical displacement of the top of the arch. For a fixed arch kinematic admissibility additionally requires that:

$$\varphi_3/\varphi_1 \geq 0 \tag{17}$$

A negative (upward) vertical displacement Δ_{ver} would result in negative external work and hence in a negative limit load. *Eqs*. (16) and (17) ensure that a negative (hogging) bending moment occurs at plastic hinge 2 and a positive (sagging) bending moment at the fixed supports, as is assumed in the bending moment distribution for the arch mechanism (see *Figs*. 2 and 3).

The vertical displacement Δ_{ver} of the top of the arch can be calculated from geometry as:

$$\Delta_{ver} = \left(\tfrac{1}{2}L - x_2\right)\varphi_2 - \tfrac{1}{2}L\varphi_3 + \delta_2 \sin\theta_2 + \delta_3 \sin\gamma \tag{18}$$

where L is the span of the arch and x_2 determines the horizontal position of plastic hinge no. 2:

$$L = 2R_n \sin\gamma \qquad x_2 = \tfrac{1}{2}L - R_n \sin\theta_2 \tag{19}$$

The contractions δ_i can be expressed in the rotations φ_i :

$$\delta_1 = c_1\varphi_1 \qquad \delta_2 = c_2\varphi_2 \qquad \delta_3 = c_3\varphi_3 \tag{20}$$

where c_1, c_2 and c_3 can be determined from the normality rule (*Eq*. 4). For the pinned arch, which has no plastic hinge at the support, c_3 should be taken equal to zero. For a fixed arch $c_3 = c_1$. Note that c_1 and c_2 depend on the normal force in the hinge. The rotation φ_3 can be determined as:

$$\varphi_3 = \varphi_2 - \varphi_1 \tag{21}$$

The rotation φ_2 can be expressed as a function of φ_1 using the condition of zero horizontal displacement Δ_{hor} of the top of the arch which may be written from geometry as [1]:

$$\Delta_{hor} = H\varphi_3 - \left(H - y_2\right)\varphi_2 + \delta_1 + \delta_2 \cos\theta_2 + \delta_3 \cos\gamma = 0 \tag{22}$$

where H is the height of the arch, and y_2 determines the vertical position of plastic hinge no. 2*:*

$$H = R_n\left(1 - \cos\gamma\right) \qquad y_2 = H - R_n\left(1 - \cos\theta_2\right) \tag{23}$$

Using *Eq*.(20) and (21) it can then be derived from *Eq*. (22) that:

$$\frac{\varphi_2}{\varphi_1} = \frac{H - c_1 + c_3 \cos\gamma}{y_2 + c_2 \cos\theta_2 + c_3 \cos\gamma} \tag{24}$$

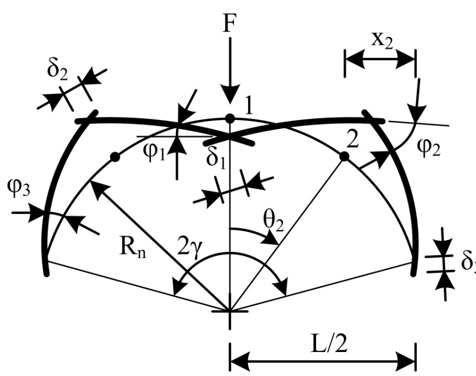
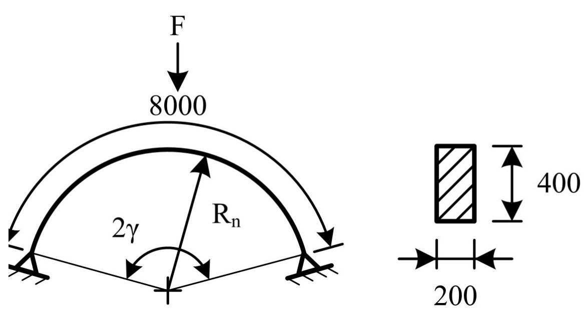


Fig. 8. Arch mechanism



(a) Sideview (b) Cross section

Fig. 9. Geometry of arches considered in parameter study

Combining *Eqs*. (21) and (24) it follows that:

$$\frac{\varphi_3}{\varphi_1} = \frac{H - y_2 - c_1 - c_2 \cos\theta_2}{y_2 + c_2 \cos\theta_2 + c_3 \cos\gamma} \tag{25}$$

Using *Eqs*. (20), (24) and (25) also the vertical displacement Δ_{ver} can be expressed as a function of φ_1. *Eqs*. (24) and (25) can be used to check whether *Eqs*. (16) and (17) are satisfied.

6 FINITE ELEMENT MODEL

To check the proposed iterative method, its results are compared to the results of geometrical linear, material non-linear finite element simulations using ANSYS release 11.0.The following input values were used (*Fig. 9*): developed length of arch S = 8000 mm, b = 200 mm, h = 400 mm. The subtended angle γ was varied. The corresponding radius R_n can be calculated as:

$$R_n = 90S/(\gamma\pi) \tag{26}$$

Bi-linear elasto-plastic material behaviour was modelled without hardening, taking f_y=235 N/mm^2 for the yield stress and E=2.1·10^5 N/mm^2 for the modulus of elasticity. In the finite element model 100, three node, curved Timoshenko beam elements (BEAM 189, [7]) were used, with 4 integration points over the height of the beam, and two groups of integration points over the length of the element (*Fig. 10*). It was found that these elements obey the following yield condition (*Fig.* 11):

$$\begin{aligned} \psi &= |M|/M_{pl} - 0.42|N|/N_{pl} - 1 = 0 \text{ if } 0 \le |N|/N_{pl} \le 0.5 \\ \psi &= |M|/M_{pl} - 1.58(1 - |N|/N_{pl}) = 0 \text{ if } 0.5 < |N|/N_{pl} \le 1 \end{aligned} \tag{27}$$

from which it can be derived that:

$$\begin{aligned} |M_{pl;red}|/M_{pl} &= 1 - 0.42|N|/N_{pl} \text{ if } 0 \le |N|/N_{pl} \le 0.5 \\ |M_{p;red}|/M_{pl} &= 1.58(1 - |N|/N_{pl}) \text{ if } 0.5 < |N|/N_{pl} \le 1 \end{aligned} \tag{28}$$

and:

$$\left|\frac{\delta}{\varphi}\right| = \left|\frac{\partial\psi/\partial N}{\partial\psi/\partial M}\right| = 0.42 \text{ if } 0 \le \frac{|N|}{N_{pl}} \le 0.5 \qquad \left|\frac{\delta}{\varphi}\right| = \left|\frac{\partial\psi/\partial N}{\partial\psi/\partial M}\right| = 1.58 \text{ if } 0.5 < \frac{|N|}{N_{pl}} \le 1 \tag{29}$$

For a better comparison between ANSYS and analytical results, *Eqs*. (28) and (29) have been used instead of *Eqs*. (3) and (4) to calculate F_{pl} and to check the kinematic admissibility of the arch mechanism.

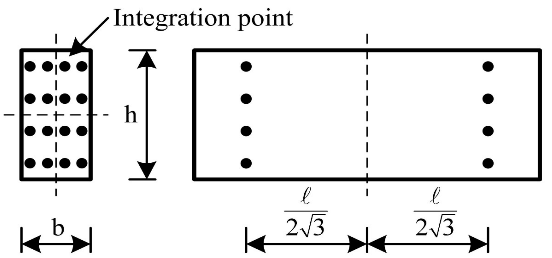


Fig. 10. Integration points for ANSYS beam element

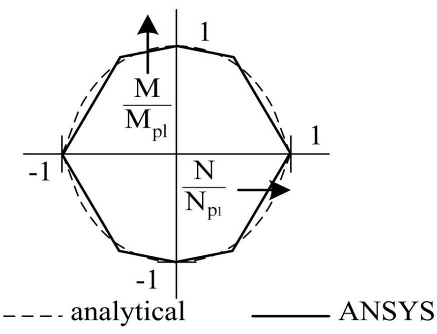


Fig. 11. Normal force and bending interaction curve for ANSYS beam element

7 RESULTS AND CONCLUSIONS

Tables 1 and 2 give an overview of the results for the pinned and fixed arch respectively. For an accuracy ε of 10^{-5} in the analytical solution (*Fig. 7*) about 20 iterations are needed. The error given in *Tables* 1 and 2 is defined as (F_{pl}-$F_{pl;ANSYS}$)/$F_{pl;ANSYS}$. It is expected that, due to the discretisation error, ANSYS will give an upper bound approximation of the limit load. It can be seen that indeed the ANSYS limit loads are always larger than the analytically determined limit loads, except for the

pinned arch with $\gamma = 5^0$. This may be caused by numerical problems: in this arch also convergence problems were encountered.
With respect to the check of the kinematic admissibility it was found that for pinned arches *Eq*. (16) is decisive, for fixed arches *Eq*. (17). The change from arch mechanism to beam mechanism occurs at $\gamma = 5^0$ for the pinned arch and at $\gamma = 35^0$ for the fixed arch. In ANSYS this transition is found to occur at $\gamma = 9^0$ respectively at $\gamma = 35^0$. As expected, the analytically determined lower bound limit loads based on the bending moment distribution of the arch mechanism become much smaller than the ANSYS limit loads, when (analytically) a beam mechanism is decisive.
It can be concluded that the proposed lower bound, iterative analytical method to determine the limit load corresponding to an arch mechanism, works well and gives good results. By checking the kinematic admissibility of the mechanism, it can be determined when a beam mechanism becomes decisive. If necessary the method can be modified to include the influence of the shear force on the reduced plastic moment capacity. The method can also be modified for use with other loading conditions and other cross-sections with their corresponding interaction curve.

Table 1. Results for pinned arch

γ [^{0}]	$F_{pl;ANSYS}$ [10^6 N]	ANSYS mech.	F_{pl} [10^6 N]	error [%]	θ_2 [^{0}]	$\frac{\varphi_2}{\varphi_1}$
1	0.942	beam	0.370	-61	0.59	<0
5	1.583	beam	1.617	+2.15	2.93	>0
10	2.845	arch	2.791	-1.90	5.87	>0
15	3.750	arch	3.680	-1.87	8.81	>0
20	4.449	arch	4.377	-1.62	11.75	>0
25	4.988	arch	4.936	-1.04	14.71	>0
30	5.227	arch	5.171	-1.07	17.61	>0
45	5.537	arch	5.466	-1.28	26.46	>0
60	5.636	arch	5.575	-1.08	35.35	>0
75	5.642	arch	5.583	-1.05	44.30	>0
90	5.576	arch	5.521	-0.99	53.32	>0

Table 2. Results for fixed arch

γ [^{0}]	$F_{pl;ANSYS}$ [10^6 N]	ANSYS mech.	F_{pl} [10^6 N]	error [%]	$\frac{\varphi_3}{\varphi_1}$
15	3.854	beam	3.482	-9.65	<0
30	5.464	beam	5.386	-1.43	<0
35	5.905	arch	5.821	-1.42	>0
45	6.242	arch	6.143	-1.59	>0
60	6.479	arch	6.372	-1.65	>0
75	6.587	arch	6.477	-1.67	>0
90	6.605	arch	6.504	-1.53	>0

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